W

Florian Kalinke

IPD PhD Colloquium, 10 November 2026

W

A continuation of counting: The Lambert W function and asymptotics

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Chance occurrence

> *First of all, denoting by W the smallest branch of the Lambert's W function on $]-e^{-1}, 0[$ (Weisstein 2002), the condition on the sub-sample size m can also be expressed as $m \geq -W(-\delta/c)c$ with $c = \max(67, 12K^2\|C\|^{-1}_{\mathcal{L}(\mathcal{H})})$ and can easily be checked on a computer.*
>
> — Chatalic et al. (2022, p. 3010)

W rediscovered

Advances in Computational Mathematics **5** (1996) 329–359

On the Lambert *W* function

R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth

University of Western Ontario · ETH Zürich · University of Waterloo · Stanford University

Abstract. The Lambert W function is defined to be the multivalued inverse of the function $w \mapsto we^w$.

W

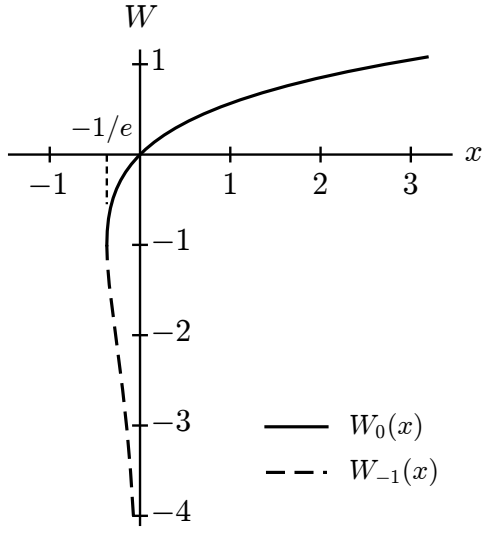
The Lambert W function is the multivalued inverse of

$$z \mapsto ze^z, \qquad z \in \mathbb{C},$$

that is,

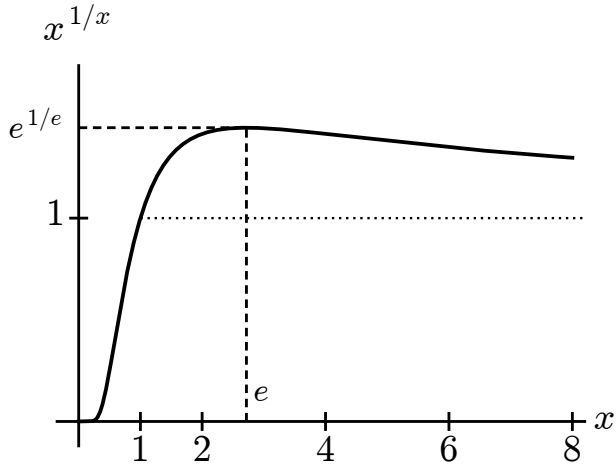
$$z = W(z)\, e^{W(z)}.$$

(Corless et al. 1996)



The two real branches of $W(x)$.

$g(x) = x^{1/x}$



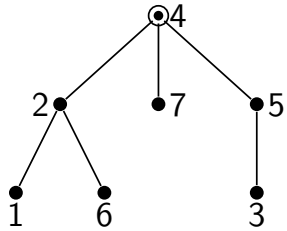
Rooted trees with labelled nodes

- t_n: number of rooted trees on n labelled nodes (Pólya 1937)

$$T(z) = \sum_{k \geq 1} t_k \frac{z^k}{k!} = z e^{T(z)}$$

- In terms of W:

$$T(z) = -W(-z)$$



Lagrange inversion formula (LIF)

Theorem (Lagrange inversion formula; LIF; Wilf 2006, Thm. 5.1.1)

Let ϕ be holomorphic in a neighborhood of the origin with $\phi(0) \neq 0$. Then there exist $a, b > 0$ such that for $|w| < a$, the equation

$$w = z/\phi(z)$$

has exactly one solution $z = z(w)$ with $|z| < b$. The function $z(w)$ is holomorphic in $|w| < a$ and is given by

$$z(w) = \sum_{k \geq 1} c_k w^k, \qquad c_k = \frac{1}{k!} \frac{d^{k-1}}{dz^{k-1}} \Big(\phi(z)^k \Big) \Big|_{z=0}.$$

Moreover, if f is holomorphic in a neighborhood of the origin, then for $|w|$ sufficiently small,

$$f\big(z(w)\big) = f(0) + \sum_{k \geq 1} d_k w^k, \qquad d_k = \frac{1}{k!} \frac{d^{k-1}}{dz^{k-1}} \Big(f'(z) \phi(z)^k \Big) \Big|_{z=0}.$$

Series expansions of T and W_0

$$t_n = n^{n-1} \qquad \text{(Cayley 1889; Pólya 1937)}$$

$$T(z) = \sum_{k\geq 1} k^{k-1}\frac{z^k}{k!} = z + z^2 + \frac{3}{2}z^3 + \frac{8}{3}z^4 + \frac{125}{24}z^5 + \cdots,$$

$$W_0(z) = \sum_{k\geq 1} (-k)^{k-1}\frac{z^k}{k!} = z - z^2 + \frac{3}{2}z^3 - \frac{8}{3}z^4 + \frac{125}{24}z^5 - \cdots$$

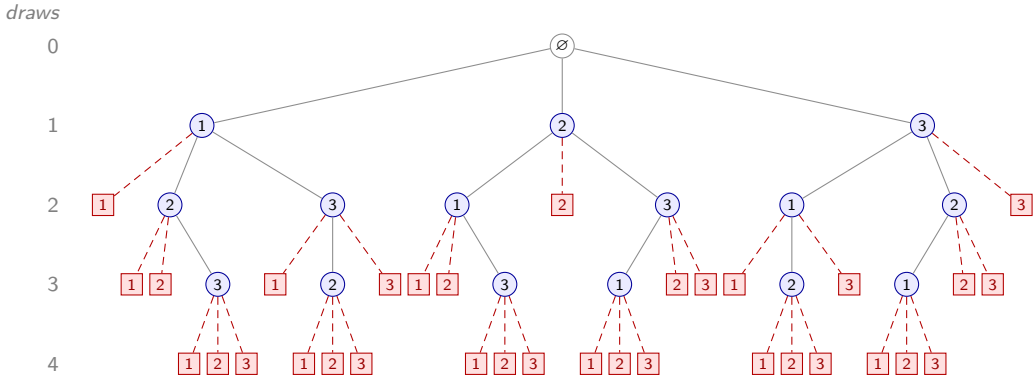
Birthday problem 1

> *What is the least number of persons required if the probability exceeds $\frac{1}{2}$ that two or more of them have the same birthday? (Year of birth need not match.)*
>
> — Mosteller (1987, Problem 31)

Birthday problem 2

> *What is the expected number of persons it takes in order to find two having the same birth date (when the year has n days!)?*
>
> — Flajolet et al. (1995, p. 105)

Birthday problem 2



Drawing with replacement from $n = 3$ items until an item repeats: ◯ new item ▭ repeated item

A helper

$$\sum_{k=m}^{N} a_k \left(b_k - b_{k+1}\right) = \left(a_m b_m - a_{N+1} b_{N+1}\right) + \sum_{k=m}^{N} b_{k+1} \left(a_{k+1} - a_k\right)$$

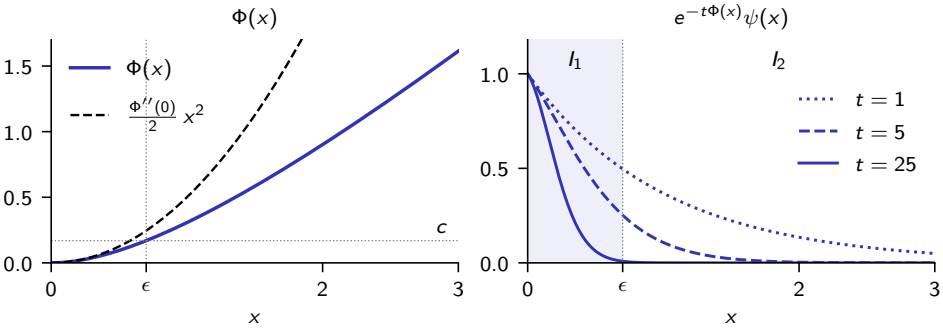
Ramanujan–Knuth Q-function

$$Q(n) = \sum_{1 \le k \le n} \frac{n!}{(n-k)!\, n^k}$$

(Knuth 1997; Flajolet et al. 1995)

Laplace method

$$I(t) = \int_0^\infty e^{-t\Phi(x)}\psi(x)\,dx \qquad (t \to \infty)$$



$$\Phi(x) = x - \ln(1+x), \quad \psi(x) = 1/(1+x)$$

What we are shooting for

With $\lambda := \Phi''(0)/2$, $u(x) := \sqrt{2\Phi(x)/\Phi''(0)}$ with inverse $x(u)$, and

$$G(u) := \psi\big(x(u)\big)\, x'(u) = \sum_{k=0}^{N-1} a_k u^k + O\big(u^N\big),$$

$$I(t) = \sum_{k=0}^{N-1} a_k \, \frac{\Gamma\left(\frac{k+1}{2}\right)}{2\,(\lambda t)^{\frac{k+1}{2}}} + O\left(t^{-\frac{N+1}{2}}\right) \qquad (t \to \infty),$$

where

$$\Gamma(z+1) = \int_0^\infty s^z e^{-s}\, ds.$$

Asymptotic analysis

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4. $Q(n) = \sqrt{\dfrac{\pi n}{2}} - \dfrac{1}{3} + \dfrac{1}{12}\sqrt{\dfrac{\pi}{2n}} + O\Big(\dfrac{1}{n}\Big) \qquad (n \to \infty)$

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5. Expected number of draws until the first repetition:

$$1 + Q(n) = \sqrt{\frac{\pi n}{2}} + \frac{2}{3} + O\Big(\frac{1}{\sqrt{n}}\Big) \qquad (n \to \infty)$$

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Appendix: Complex analysis

Theorem

A holomorphic function in an open disc has a primitive in that disc.

Appendix: Complex analysis

Theorem

A holomorphic function in an open disc has a primitive in that disc.

Theorem (Cauchy's theorem for a disc)

If f is holomorphic in a disc, then

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve γ in that disc.

Appendix: Complex analysis

Theorem (Cauchy integral formula)

Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \text{for any point } z \in D.$$

Appendix: Complex analysis

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Corollary

If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

for all z in the interior of C.

Appendix: Complex analysis

Theorem (Residue formula)

Suppose that f is holomorphic in an open set containing a circle C and its interior, except for a pole at z_0 inside C. Then

$$\int_C f(z)\,dz = 2\pi i \operatorname{res}_{z_0} f.$$