Counting Tree Structures

Generating functions, abstract algebra, and PET

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Guiding questions

- Given three distinct dice, what is the probability that their eyes sum to ten?
- How many ways are there of making change for $1?
- What is the number of necklaces that can be constructed from five beads available in three distinct colors?
- In how many ways can the corners of a square be colored?
- How many trees with a given number of nodes are there?

Generating functions

- Encode sequences, e.g., $\langle g_n \rangle$.
- Ordinary generating function (ogf):[1]

$$G(z) = g_0 + g_1 z + g_2 z^2 + \cdots = \sum_{n \geq 0} g_n z^n, \quad z \in \mathbb{C}.$$

- n-th coefficient:

$$[z^n] G(z) := g_n.$$

[1] An ogf is not (necessarily) a function nor does it generate anything.

Summing dice

Given three distinct dice, what is the probability that their eyes sum to ten?

Calculating change

How many ways are there of making change for $1?

- Perspective 1: Cauchy product.
- Perspective 2: Universe of distinct structures and weight function.

Counting necklaces: issue

What is the number of necklaces that can be constructed from five beads available in three distinct colors?

Counting necklaces: issue

What is the number of necklaces that can be constructed from five beads available in three distinct colors?

- Trivial answer: 5^3.

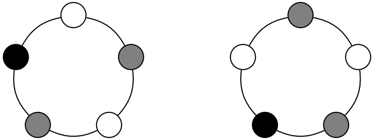

Figure: Two essentially equivalent necklaces but both accounted for by the trivial answer.

Counting necklaces: symmetries

Lemma (Frobenius, 1887; Burnside, 1897)

Let G be a group acting on the set S and $\chi(g)$ the number of elements fixed by $g \in G$. Then

$$\frac{1}{|G|}\sum_{g\in G}\chi(g) = \textit{the number of orbits of } S \textit{ under } G.$$

Counting necklaces: symmetries

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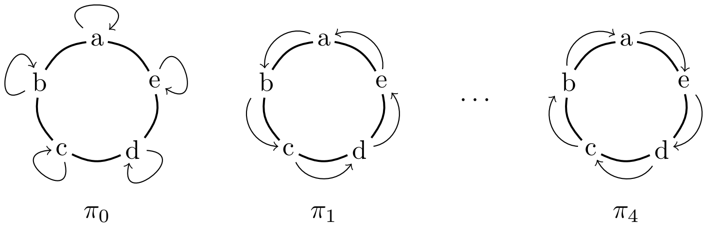

Figure: The rotations π_0 through π_4 that act on the necklace.

Counting square colorings: symmetries

How many 2-colorings of the corners of a square are there up to rotations and reflections?

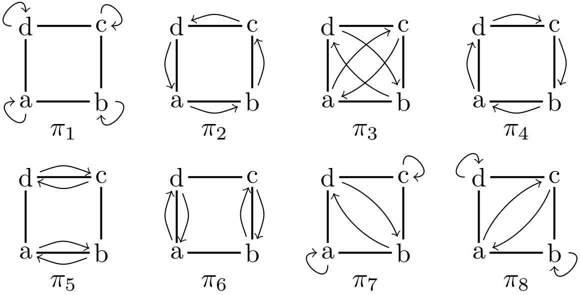

Figure: The rotations and reflections of a square.

Counting square colorings: all colorings

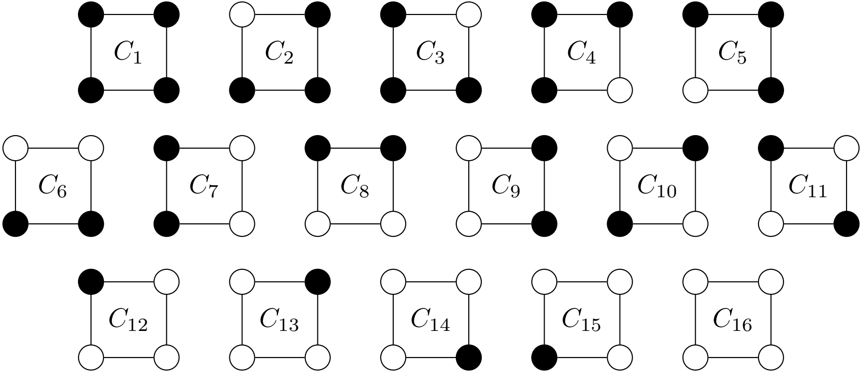

Figure: The different colorings of a fixed square.

PET (Pólya enumeration theorem)

How many colorings with a given number of black and white beads are there if we identify colorings as essentially equivalent that match up to rotations and reflections?

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Theorem (Pólya, 1937)

The configuration counting series $C(z)$ is determined by substituting for each variable z_j in $Z(G; z_1, z_2, \ldots)$ the figure counting series $c(z^j)$. Symbolically,

$$C(z) = Z\big(G; c(z), c(z^2), \ldots\big) =: Z(G; c(z)).$$

Counting trees: tree types

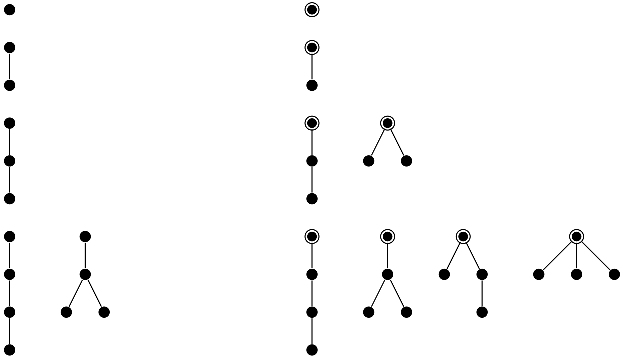

Figure: Free trees (left) and rooted trees (right) ordered by node count.

Counting trees: rooted trees

Figure: Combining four rooted trees to a rooted tree with root degree four.

Counting trees: rooted trees

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Theorem

The configuration counting series $T(z)$ for rooted trees satisfies

$$T(z) = z \exp\left\{ \sum_{k=1}^{\infty} T\left(z^k\right)/k \right\}.$$

Counting trees: Otter's theorem

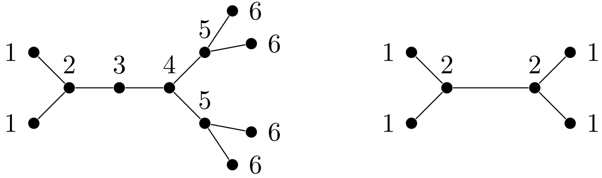

Figure: Two trees and the equivalence classes of their points. The left tree has no symmetry line while the right tree has a symmetry line.

Counting trees: Otter's theorem

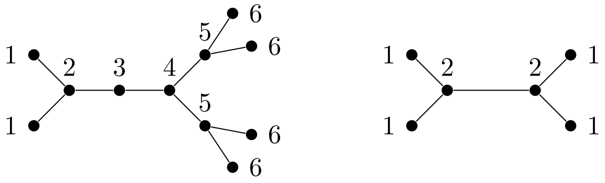

Figure: Two trees and the equivalence classes of their points. The left tree has no symmetry line while the right tree has a symmetry line.

Theorem (Otter, 1948)

The number s of symmetry lines of any tree is 0 or 1 and

$$p^* - (q^* - s) = 1.$$

Counting trees: another result of Pólya

Theorem (Pólya 1937)

The generating function $C(z)$ that enumerates one-to-one functions from n indistinguishable elements into a collection of objects with figure counting series $c(z)$ is given by

$$C(z) = Z(A_n; c(z)) - Z(S_n; c(z)) =: Z(A_n - S_n; c(z)),$$

where A_n is the alternating and S_n the symmetric group.

Counting trees: unrooted trees

Figure: Two rooted trees and the corresponding line-rooted tree.

Counting trees: unrooted trees

Figure: Two rooted trees and the corresponding line-rooted tree.

Theorem

The counting series $t(z)$ for unrooted trees is expressed in terms of the counting series of rooted trees $T(z)$ as

$$t(z) = T(z) - \frac{1}{2}\Big(T^2(z) - T\left(z^2\right)\Big).$$

References I

W. Burnside. *Theory of Groups of Finite Order*. Cambridge University Press, 1897.

G. Frobenius. Ueber die Congruenz nach einem aus zwei endlichen Gruppen gebildeten Doppelmodul. *Journal für die Reine und Angewandte Mathematik*, 101:273–299, 1887.

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G. Pólya. Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Mathematica*, 68(1):145–254, 1937.