# Nyström Kernel Stein Discrepancy*

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## Quick Summary


- Kernel Stein discrepancy (KSD; [1, 2]): powerful goodness-of-fit measure and test.
- Applications: Assessing and improving sample quality, validating MCMC methods, comparing deep generative models, ....
- Limitations: Quadratic runtime complexity.
- Main contribution: Accelerated estimator with the same convergence rate as the quadratic-time estimator.


## Kernel Stein Discrepancy

- Goal: Test $H_0 : \mathbb{P} = \mathbb{Q}$ vs. $H_1 : \mathbb{P} \neq \mathbb{Q}$ for fixed known target $\mathbb{P}$ and unknown sampling distribution $\mathbb{Q}$, given samples $\hat{\mathbb{Q}}_n := \{\mathbf{x}_i\}_{i=1}^n \subset \mathbb{R}^d$ of $\mathbb{Q}$.
- We illustrate the method with the Langevin-Stein operator-based [3] KSD on $\mathbb{R}^d$, which is

$$S_p(\mathbb{Q}) = \left\| \mathbb{E}_{X\sim\mathbb{Q}} h_p(\cdot, X) \right\|_{\mathcal{H}_{h_p}},$$

with kernel ($\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$)

$$\begin{aligned} h_p(\mathbf{x},\mathbf{y}) := & \left\langle \nabla_{\mathbf{x}} \log p(\mathbf{x}), \nabla_{\mathbf{y}} \log p(\mathbf{y}) \right\rangle_{\mathbb{R}^d} k(\mathbf{x},\mathbf{y}) \\ & + \left\langle \nabla_{\mathbf{y}} \log p(\mathbf{y}), \nabla_{\mathbf{x}} k(\mathbf{x},\mathbf{y}) \right\rangle_{\mathbb{R}^d} \\ & + \left\langle \nabla_{\mathbf{x}} \log p(\mathbf{x}), \nabla_{\mathbf{y}} k(\mathbf{x},\mathbf{y}) \right\rangle_{\mathbb{R}^d} + \sum_{i=1}^d \frac{\partial^2 k(\mathbf{x},\mathbf{y})}{\partial x_i \partial y_i}, \end{aligned}$$

$p$ the (Lebesgue) density of $\mathbb{P}$, and kernel $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$.
- Note that knowledge of the derivative of the score function $\nabla_{\mathbf{x}} \log p(\mathbf{x})$ is enough, i.e., knowledge of $p$ up to normalization suffices.
- Existing KSD estimators take (roughly) the form

$$S_p^2\left(\hat{\mathbb{Q}}_n\right) = \frac{1}{n^2} \sum_{i,j=1}^n h_p(\mathbf{x}_i, \mathbf{x}_j),$$

and have a runtime requirement of $\mathcal{O}(n^2)$.

## Nyström-based Estimator (N-KSD)

- Denote by $\tilde{\mathbb{Q}}_m := \{\{\tilde{\mathbf{x}}_i\}\}_{i=1}^m$ a subsample of $\hat{\mathbb{Q}}_n$. The Nyström estimator is

$$\tilde{S}_p^2\left(\hat{\mathbb{Q}}_n\right) = \boldsymbol{\beta}_p^\mathsf{T} \mathbf{K}_{h_p,m,m}^- \boldsymbol{\beta}_p,$$

with $\boldsymbol{\beta}_p = \frac{1}{n}\mathbf{K}_{h_p,m,n}\mathbf{1}_n \in \mathbb{R}^m$, matrices

$$\begin{aligned} \mathbf{K}_{h_p,m,m} &= \left[h_p(\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}_j)\right]_{i,j=1}^m \in \mathbb{R}^{m\times m}, \text{ and} \\ \mathbf{K}_{h_p,m,n} &= \left[h_p(\tilde{\mathbf{x}}_i, \mathbf{x}_j)\right]_{i,j=1}^{m,n} \in \mathbb{R}^{m\times n}, \end{aligned}$$

and $\mathbf{A}^-$ denoting the (Moore-Penrose) pseudo-inverse of a matrix $\mathbf{A}$.
- Runtime: $\mathcal{O}(mn + m^3)$, saving if $m = o\left(n^{2/3}\right)$.

## Sub-Gaussian Assumption

- Existing Nyström analysis considers bounded kernels only. In practice, $h_p$ is usually unbounded and existing results do not apply.
- Example: Consider $d = 1$, standard normal $p(x) \propto \exp(-x^2/2)$, and the RBF kernel $k(x, y) = \exp(-\gamma(x-y)^2)$ ($\gamma > 0$). Then

$$h_p(x, x) = x^2 + 2\gamma \overset{x\to\infty}{\to} \infty.$$

Similarly, for the IMQ kernel $k(x, y) = (c^2 + (x-y)^2)^{-\beta}$ ($\beta, c > 0$).
- For the Nyström analysis, assume that $\bar{h}_p(\cdot, X) := h_p(\cdot, X) - \mathbb{E}_{X\sim\mathbb{Q}} h_p(\cdot, X)$ with the sampling distribution $\mathbb{Q}$ is sub-Gaussian, that is,

$$\left\| \left\langle \bar{h}_p(\cdot, X), u \right\rangle_{\mathcal{H}_{h_p}} \right\|_{\psi_2} \lesssim \left\| \left\langle \bar{h}_p(\cdot, X), u \right\rangle_{\mathcal{H}_{h_p}} \right\|_{L_2(\mathbb{Q})} < \infty \qquad (1)$$

holds for all $u \in \mathcal{H}_{h_p}$, with a $u$-independent absolute constant in $\lesssim$, and $\|\cdot\|_{\psi_2}$ denoting the sub-Gaussian norm.

## Main Results

- $\sqrt{n}$-consistency of KSD estimator: If

$$\left\| \left\| h_p(\cdot, X) \right\|_{\mathcal{H}_{h_p}} \right\|_{\psi_2} < \infty$$

holds (implied by (1)), then

$$\left| S_p(\mathbb{Q}) - S_p\left(\hat{\mathbb{Q}}_n\right) \right| = \mathcal{O}_P\left(n^{-1/2}\right).$$

- $\sqrt{n}$-consistency of N-KSD: If the sub-Gaussian property (1) holds, then

$$\left| S_p(\mathbb{Q}) - \tilde{S}_p\left(\hat{\mathbb{Q}}_n\right) \right| = O_P\left(n^{-1/2}\right),$$

given that the effective dimension $\mathcal{N}_{\mathbb{Q},\bar{h}_p}(\lambda) := \operatorname{tr}\left(C_{\mathbb{Q},\bar{h}_p,\lambda}^{-1} C_{\mathbb{Q},\bar{h}_p}\right)$ ($C_{\mathbb{Q},\bar{h}_p} := \mathbb{E}_{X\sim\mathbb{Q}}[h_p(\cdot, X) \otimes h_p(\cdot, X)]$; $C_{\mathbb{Q},\bar{h}_p,\lambda} := C_{\mathbb{Q},\bar{h}_p} + \lambda I$, $\lambda > 0$) either
  - decays polynomially:

$$\mathcal{N}_{\mathbb{Q},\bar{h}_p}(\lambda) \lesssim \lambda^{-\gamma}, \qquad m = \tilde{\Omega}\left(n^{1/(2-\gamma)}\right),$$

  for $\gamma \in (0, 1]$ (computational savings if $\gamma < 1/2$), or
  - decays exponentially:

$$\mathcal{N}_{\mathbb{Q},\bar{h}_p}(\lambda) \lesssim \log(1 + c_1/\lambda), \qquad m = \tilde{\Omega}\left(n^{1/2}\right),$$

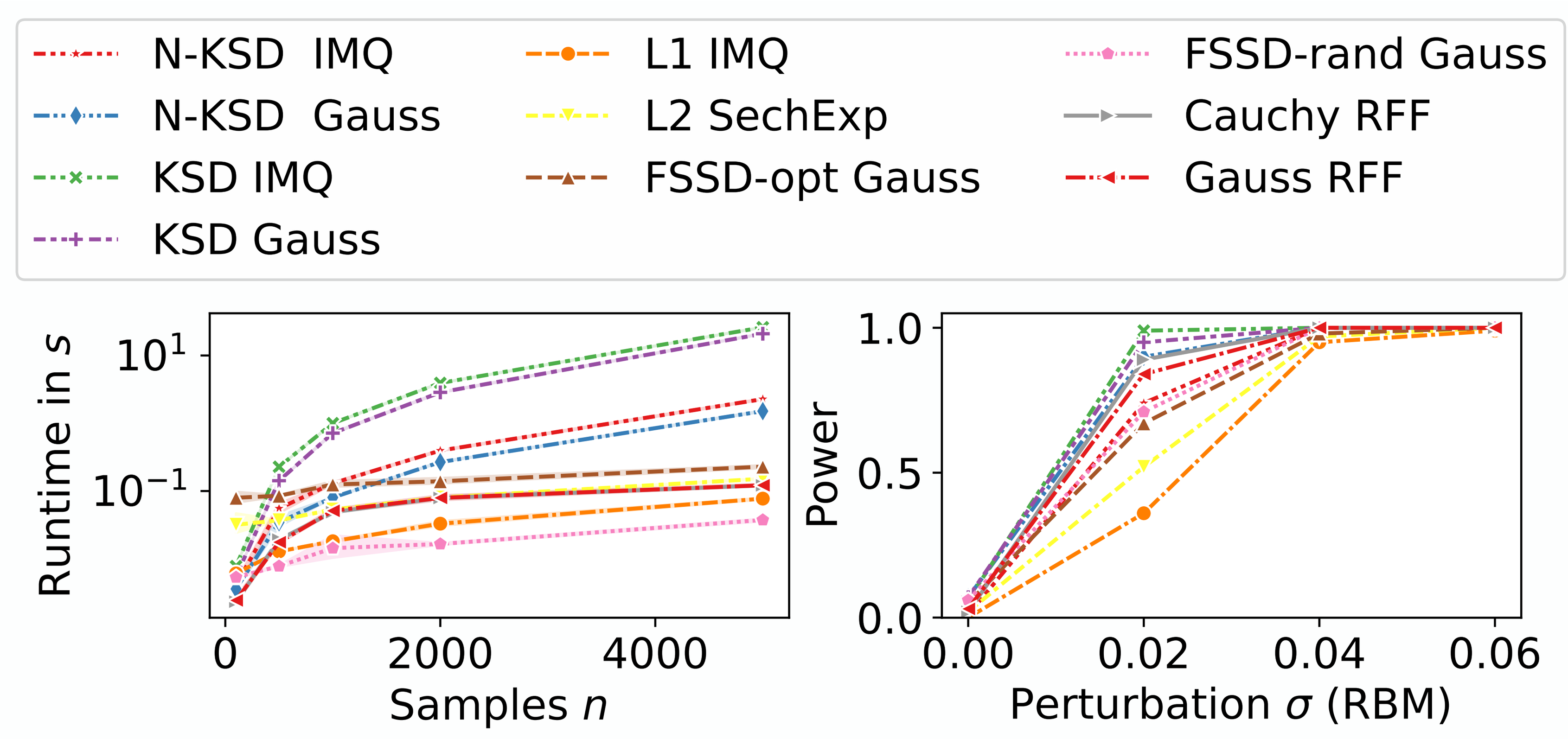
  for some $c_1 > 0$ (computational savings if $n$ is large enough).
- The decay of the effective dimension can be linked to the decay of the eigenvalues of the covariance operator $C_{\mathbb{Q},\bar{h}_p}$ [4, Proposition 4, 5].

## Discussion

- Unboundedness of the feature map handled by sub-Gaussian assumption.
- The quadratic-time and the N-KSD estimator both have $\sqrt{n}$-consistency, i.e., computational gain with no loss in statistical accuracy.
- Our results apply in the general KSD framework [5].
- Open: Weaker assumption for the Nyström case.

## Goodness-of-fit Benchmark

- Runtime and power of Nyström KSD (N-KSD) and competitors on the restricted Boltzmann machine (RBM) goodness-of-fit benchmark.




- Code: `https://github.com/flopska/nystroem-ksd`

## References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